# Remarks about the thermostatistical description of the HMF model Part II: Phenomenology of Relaxation Dynamics

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## Abstract

After a general overview of some features of the relaxation dynamics of the Hamiltonian Mean Field model, its equilibrium thermodynamic properties are used to rephrase the out-of-equilibrium regime for energies below the critical point $u_c = 0.75$ in terms of an effective dynamical coexistence between a clustered and a gaseous phases, whose existence could be associated to the large relaxation times observed when $u_1 < u < u_c$, with $u_1 = 0.5$. Starting from the hypothesis that the *parametric resonance* is the microscopic mechanism allowing the energetic interchange between the particles during the collisional regime, a phenomenological Fokker-Planck equation based on a Langevin equation with a multiplicative noise is proposed in order to describe the collisional relaxation of this system towards its final equilibrium, which supports the following dependence of the collisional relaxation timescale $\tau_{cr} = \tau_0 N \equiv \sqrt{IN/g}$.

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## I. INTRODUCTION

Despite of the Hamiltonian Mean Field (HMF) model [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18] is a ferromagnetic toy model enough amenable for allowing an accurate numerical and analytical characterization, it exhibits many features observed in more realistic long-range interacting systems such as: violent relaxation, persistence of metaequilibrium states, slow collisional relaxation, phase transition, anomalous diffusion, etc. These features support the reason why it can be considered as a paradigmatic toy model of the real long-range interacting systems [14].

The incidence of long-range interactions does not allow to divide this system into independent subsystems even in the thermodynamic limit. This fact evidences its intrinsic nonextensive nature, which distinguishes the HMF model from other ferromagnetic models with short-range interactions despite they share many analogies in most of the equilibrium thermodynamic properties [14]. The nonextensivity of the HMF model is particularly important to understand its nontrivial dynamical behavior, which is still an open problem attracting much attention in the last years [11, 12, 13, 14, 15, 16, 17, 18].

The present paper of this series [19] is the first work devoted to study the dynamical behavior of the HMF model with a special emphasis on the relaxation processes undergone by this system during the evolution towards the thermodynamic equilibrium. We begin performing a general characterization of the most important features of the microscopic dynamics by taking into consideration the results obtained from the analysis of its equilibrium thermodynamics [19]. Such a study possibilities us to conjecture the equilibration mechanism leading this system to its final relaxation, which is here used to propose a phenomenological approach of the collisional evolution.

## II. GENERAL CHARACTERIZATION

### A. Numerical computation of microscopic dynamics and the Vlasov equation

According to the Hamiltonian:

$$H_N = \sum_{i=1}^{N} \frac{1}{2I} L_i^2 + \frac{1}{2} g \sum_{i=1}^{N} \sum_{j=1}^{N} \left[1 - \cos\left(\theta_i - \theta_j\right)\right], \quad (1)$$

the motion equations of the HMF model are given by:

$$\dot{\theta}_i = \frac{1}{I} L_i, \;\; \dot{L}_i = gN \left(m_y \cos\theta_i - m_x \sin\theta_i\right), \quad (2)$$

for $i = 1, 2, ...N$, where $m_x$ and $m_y$ are the Cartesian components of the magnetization vector $\mathbf{m} = (m_x, m_y) = \left(\sum_{i=1}^{N} \mathbf{m}_i\right)/N$ where $\mathbf{m}_i = (\cos\theta_i, \sin\theta_i)$.

From a dynamical viewpoint, the HMF model represents the long-range interacting version of harmonic oscillators system [14] since the individual dynamics of a given rotator can be easily rephrased as the dynamics of the *mathematical pendulum*:

$$\ddot{\theta}_i + m \sin\left(\theta_i - \theta\right) = 0. \quad (3)$$

We have used the polar representation of the magnetization vector $\mathbf{m} = m\left(\cos\theta, \sin\theta\right)$ and the characteristic units for time and momentum:

$$\tau_0 = \sqrt{\frac{I}{gN}} \text{ and } L_0 = \sqrt{IgN}, \quad (4)$$

in terms of the moment of inertia $I$ and the coupling constant $g$. Obviously, the characteristic time unit $\tau_0$

provides the timescale for the evolution of each rotator, and hence, $\tau_0$ is the characteristic microscopic timescale. The use of characteristic unit $L_0$ allows to express the total energy per particle $u = k + \frac{1}{2}\left(1 - \mathbf{m}^2\right)$ and the kinetic energy per particle $k = K/N = \left(\sum_{i=1}^{N} \frac{1}{2} p_i^2\right)/N$, with $p_i = L_i/L_0$, in terms of the characteristic unit $\varepsilon_0 = E_0/N = gN$ introduced in our previous paper [19].

Another convenient way to rewrite the dynamics of the HMF model (2) follows from the introduction of the tridimensional vectors $\mathbf{m}_i = (\cos\theta_i, \sin\theta_i, 0)$ and $\mathbf{o}_k = (0, 0, p_i)$:

$$\dot{\mathbf{m}}_i = \mathbf{o}_i \times \mathbf{m}_i, \;\; \dot{\mathbf{o}}_i = \mathbf{m}_i \times \mathbf{m}. \tag{5}$$

where the magnetization vector $\mathbf{m}$ obeys the following dynamics:

$$\dot{\mathbf{m}} = \boldsymbol{\Upsilon} = \frac{1}{N} \sum_{i=1}^{N} \mathbf{o}_i \times \mathbf{m}_i, \tag{6}$$

being $\boldsymbol{\Upsilon}$ the magnetization rate of change.

A simple inspection of Eqs.(3), (5) and (6) allows to understand that the dynamical phenomenology the HMF model can be explained from the conjugation of the dynamical features of the mathematical pendulum and the character of the magnetization vector $\mathbf{m}$ evolution. Since the individual terms $\mathbf{o}_k \times \mathbf{m}_k$ in the magnetization rate of change $\boldsymbol{\Upsilon}$ have a undefined signature, the dynamical evolution of magnetization vector crucially depends on the correlations among the individual rotators evolution. If present, the magnetization experiences large variations in a timescale comparable with the microscopic time $\tau_0$, where takes place a very effective energy interchange among the rotators. This dynamical regime characterized by the existence of a collective motion completely analogue to the one operating in the astrophysical systems, which is usually referred as *violent relaxation* [20] and very-well described by the *Vlasov dynamics*:

$$\left(\frac{\partial}{\partial t} + p\frac{\partial}{\partial \theta} + \mathbf{m}(\theta) \times \mathbf{m}[f] \frac{\partial}{\partial p}\right) f(\theta, p; t) = 0, \tag{7}$$

with $\mathbf{m}[f] = \int \mathbf{m}(\theta) f(\theta, p; t)\, d\theta dp$, which accounts the *collisionless* regime of the one-body distribution function $f(\theta, p; t)$. This fast relaxation regime finishes after arriving at certain stable *quasi-stationary state* (QSS) of Eq.(7), $f(\theta, p; t) \to f_{QSS}(\theta, p) = F[\varepsilon(p, \theta)]$, where $\varepsilon(p, \theta) = \frac{1}{2}p^2 - \mathbf{m} \cdot \mathbf{m}(\theta)$ is the energy of an individual rotator. Hereafter, the dynamical evolution depends on *collisional effects that* decrease with the system size $N$, and hence, the system evolves throughout stable QSSs towards its final equilibration [11, 12, 13, 14, 15, 16, 17, 18].

During this quasi-stationary regime, the individual rotators becomes almost non correlated, since the two-body correlation function probably survives at the $1/N$ approximation as its microcanonical estimate, Eq.(58) of ref.[19]. Therefore, their short-time dynamics can be described as free motions of a mathematical pendulum. The vector $\boldsymbol{\Upsilon}$ experiences zero mean fluctuations in the microscopic timescale $\tau_0$ whose amplitudes decrease with the system size as $1/\sqrt{N}$, provoking in this way a very small fluctuating behavior of the magnetization vector $\mathbf{m}$ around certain mean value. Such fluctuations constitute the microscopic relaxation mechanism allowing the energetic interchange among the rotators, which reduces its effectiveness with the increasing of the system size $N$. This last observation clarifies the reason why the characteristic relaxation timescale of such a collisional regime $\tau_{cr} \sim \tau_{eq}$ grows with $N$.

The above picture is easily verified by means of microcanonical numerical computation of dynamics (2). For comparison purposes, we also carry out the numerical integration of the collisionless dynamics (7) by imposing a truncation of distribution function $f(\theta, p; t) \equiv 0$ for $|p| \geq p_c$ and introducing a second-order finite differences scheme as follows:

$$\frac{\partial f_{ij}}{\partial t} + p_j \Delta_\theta f_{ij} + \mathbf{m}_i \times \mathbf{m}[f] \Delta_p f_{ij} = 0, \tag{8}$$

where $f_{i,j} = f(\theta_i, p_j)$ with $\theta_i = ih_1$ and $p_j = jh_2$, being $i = [1, 2, \ldots, N_1 + 1]$, $j = [-N_2 - 1, \ldots, N_2 + 1]$, $h_1 = 2\pi/(N_1 + 1)$ and $h_2 = p_c/(N_2 + 1)$, with the boundary conditions:

$$f_{N_1+1,j} = f_{1,j}, \quad f_{i,\pm(N_2+1)} = 0. \tag{9}$$

Besides, magnetization is rephrased as $\mathbf{m}[f] = h_1 h_2 \sum_{ij} \mathbf{m}_i f_{ij}$ with $\mathbf{m}_i = \mathbf{m}(\theta_i)$, and the partial numerical derivatives are given by $\Delta_\theta f_{ij} = (f_{i+1,j} - f_{i,j})/h_1$ and $\Delta_p f_{ij} = (f_{i,j+1} - f_{i,j-1})/2h_2$. Numerical integrations of the microscopic dynamics (2) and the scheme (8) with $N_1 = 101$ and $N_2 = 200$ where performed by using fourth-order Runge-Kutta method with constant timesteps $\delta t_m = 0.05$ and $\delta t_v = 0.02$ respectively, which ensure a good enough conservation of the energy per particle $u$. All microscopic simulations starts from *water bag initial conditions* (WB): $|\mathbf{m}(t = 0)| = 1$ $(\forall i,\ \theta_i = 0)$ with a random uniform distribution for rotators momenta $[-p_*, p_*]$ with $p_* = \sqrt{6u}$, an unstable initial condition which can be considered as microscopic configuration highly ordered. In order to avoid abrupt changes in the partial numerical derivatives $\Delta_\theta f_{ij}$ and $\Delta_p f_{ij}$, the WB initial conditions of the microscopic dynamics were approximated by the following initial condition for the one-body distribution function:

$$f_{wb}(\theta, p) \propto \frac{1}{1 + \exp[\lambda_\theta(|\theta| - \theta_*)]} \frac{1}{1 + \exp[\lambda_p(|p| - p_*)]},$$

with $\lambda_\theta = \lambda_p = 50$ and momentum cutoff $p_c = 1.5p_*$. The small parameter $\theta_*$ used here ensures a precision of the initial magnetization as $\delta|\mathbf{m}(0)| \sim 10^{-3}$. As in many studies [7, 8, 10, 11, 12, 13], the present calculations were performed at $u = 0.69$, an energy value corresponding at the equilibrium to the ferromagnetic state close to the

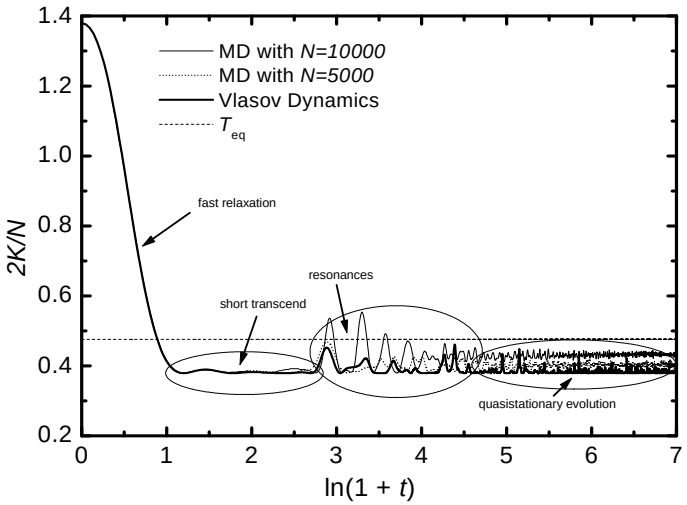


FIG. 1: Evolution of the nonequilibrium temperature $T_D$ obtained from the microscopic dynamics (MD) with $N = 5000$ and 10000, which can be compared to the Vlasov collisionless dynamics. Horizontal scale represents the function $\ln(1+t)$ in order to exhibit the whole simulation time interval by using a linear scale for small $t$ and in logarithmic scale when $t \gg 1$.

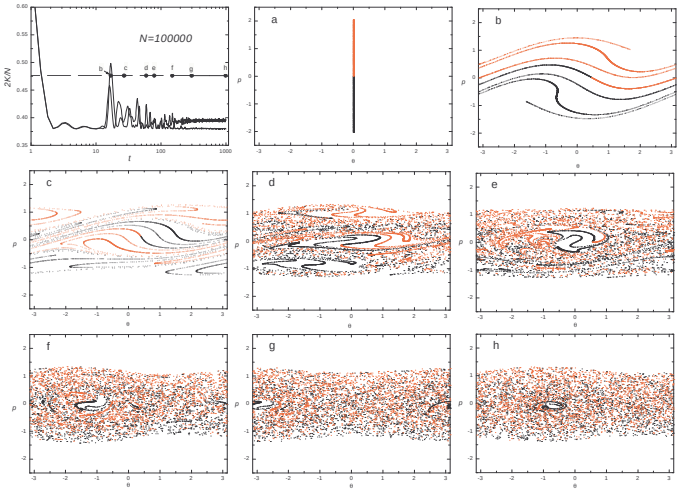


FIG. 2: First panel: Dynamical evolution of the nonequilibrium temperature associated with two realizations of the wb initial conditions with $N = 100000$, where it could be appreciated a dynamical sensibility to the initial conditions. Panels (a-h) display different instants (indicated in the first panel) of the dynamical evolution of the system in the $\mu$-space, where it is revealed the formation and destruction of filamentary structures. Black and red points are use to illustrate the mixing of microscopic dynamics.

critical point $u_c = 0.75$ of the continuous phase transition.

FIG.1 shows the evolution of the *nonequilibrium temperature* $T_D = 2K/N$ (two times the kinetic energy per particle) obtained from two individual trajectories of (2) with $u = 0.69$ for $N = 5000$ and 10000 respectively. Notice that the temporal dependences of $T_D$ derived from the microscopic dynamics can be compared to the evolution of this same quantity obtained from the numerical implementation of the Vlasov dynamics.

Size effects are hardly appreciated at the first stages of the microscopic dynamics where takes place a very fast relaxation followed by a short transcend regime (the plateau in $T_D$), which finishing with another relaxation regime where take place the development of resonances (peaks) with the incidence of finite size effects. The fact that the immediate response of the system can be accounted for by the Vlasov dynamics demonstrates the existence of a collisionless evolution during this initial relaxation regime, whose relaxation time $\tau_{vr}$ is comparable to the microscopic timescale $\tau_0$, $\tau_{vr} \sim \tau_0$. FIG.1 also shows that the size effects and fluctuations are very important in the subsequent evolution after the violent relaxation. The large transcend regime evidences the occurrence of quasi-stationary evolution where the nonequilibrium temperature $T_D$ remains a long time fluctuating around a nonequilibrium value (compare it with the equilibrium temperature $T_{eq} = 0.476$ represented as a horizontal line).

The evolution of a given realization of the microscopic dynamics in the $\mu$-space $(\theta, p)$ is shown in FIG.2 for $N = 100000$. It illustrates how the highly ordered initial state is progressively destroyed by the incidence of chaoticity and mixing of the collisionless regime. Such mechanisms manifest in the development of filamentary structures, which disappear at a coarsed grained scale leading thus to the establishment of the stable QSS. This process is also displayed in FIG.3, but this time, the evolution of the one-body distribution function obtained from the finite differences scheme (8) where it could be also noticed the formation and destruction of filamentary structures.

Such a large transcend regime could be identify with the presence of a stable QSS of the Vlasov dynamics (7), which actually evolves in the collisional timescale $\tau_{cr} \gg \tau_0$. This fact has been shown throughout extensive calculation of the average of the nonequilibrium temperature $T_D$ with $N = 10^2 - 10^4$ rotators over $\sim 10^3 - 10^4$ realizations of microscopic dynamics, where it is shown a growing of the relaxation timescale $\tau_{eq}$ with $N$ [11, 12, 13]. As already pointed out theoretically and observed in many numerical studies, the nature and the relaxation time of such a collisional quasi-stationary evolution depend on the initial conditions.

The qualitative features of system evolution are also captured by the *diffusion rate* $\omega(t) = d\Delta_\theta(t)/dt$ defined from the square dispersion $\Delta_\theta(t)$:

$$\Delta_\theta(t) = \frac{1}{2N}\sum_{i=1}^{N}\left[\theta_i(t) - \theta_i(0)\right]^2, \qquad (10)$$

which provides a measure of the average correlation func-

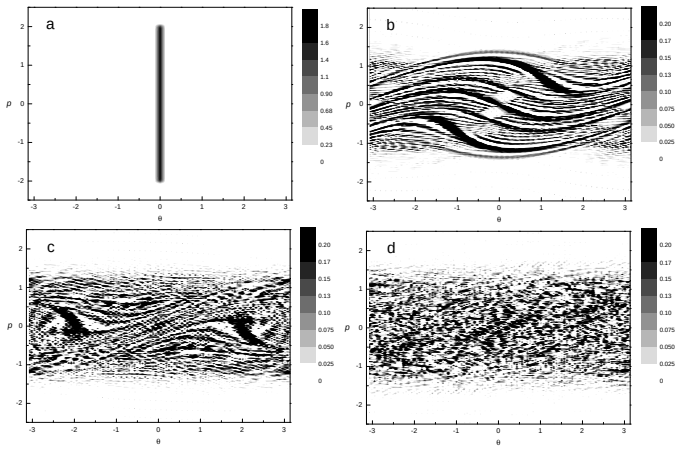


FIG. 3: Collisionless evolution of one-body distribution function obtained from the numerical integration of the finite differences scheme (8) where it can be also appreciated the formation and destruction of filamentary structures. We illustrate here the first four instants shown in the FIG.2.

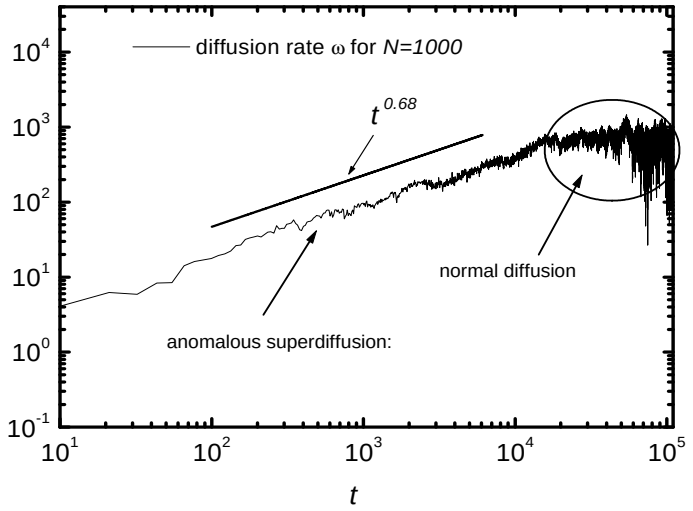


FIG. 4: Evolution of diffusion rate $\omega(t)$ obtained from the numerical simulation of the microscopic dynamics with $N = 1000$, which evidences the existence of a crossover time at $\tau_{cross} \sim 10^4$ where the anomalous superdiffusion with effective exponent $\alpha \simeq 1.68$ turns a normal diffussion.

tion:

$$\omega(t) = \frac{1}{N}\sum_{i=1}^{N}\omega_i(t)\,, \text{with } \omega_i(t) \equiv \int_0^t L_i(t)\,L_i(t-\tau)\,d\tau, \tag{11}$$

and accounts for the character of the diffusion regimes undergone by the system[1]. Dynamical studies revealed that after a brief ballistic regime with $\alpha = 2$, the system exhibits an *anomalous superdiffusion* during the quasi-stationary evolution. The long time calculation of the diffusion rate illustrated in FIG.4 allows to estimate the exponential constant $\alpha \simeq 1.68$ for the leading behavior of the square dispersion with $N = 1000$. Notice the existence of a crossover time $\tau_{cross} \sim 10^4$ where the diffusion regime turns *normal*, which can be related to the final evolution of the collisional regime towards the achievement the of the Boltzmann-Gibbs equilibrium. Despite the incidence of size effects in the crossover time $\tau_{cross}$ and the exponent $\alpha$, the general behavior of the diffusional regime described above is consistent for different system sizes $N$.

### B. Relaxation of two–body correlation function

By denoting $\delta[x - x_i(t)] \equiv \delta[\theta - \theta_i(t)]\,\delta[p - p_i(t)]$ with $x = (\theta, p)$ and the average over many realizations of the microscopic dynamics $\langle\cdot\rangle$, we are able to introduce the one-body and the two-body distribution functions:

$$f(x;t) = \left\langle \tfrac{1}{N}\textstyle\sum_{i=1}^{N}\delta[x - x_i(t)]\right\rangle, \tag{12}$$

$$f^{(2)}(x_1, x_2; t) = \tfrac{2}{N(N-1)}\left\langle \textstyle\sum_{i>j}^{N}\delta[x_1 - x_i(t)]\,\delta[x_2 - x_j(t)]\right\rangle,$$

as well as the two-body correlation function $g(x_1, x_2; t)$ as follows:

$$f^{(2)}(x_1, x_2; t) = f(x_1; t)\,f(x_2; t) + g(x_1, x_2; t)\,. \tag{13}$$

These distributions functions can be used to define the average $\langle a\rangle$ and the correlation function $c_a$ of a microscopic observable $a(x) = a(\theta, p)$ of an individual rotator:

$$\langle a\rangle \equiv \int a(x)\,f(x;t)\,dx, \tag{14}$$

$$c_a = \int a(x_1)\,a(x_2)\,g(x_1, x_2; t)\,dx_1 dx_2, \tag{15}$$

with $dx = d\theta dp$, which represent two indicators of their corresponding dynamical evolutions.

The correlation function of the kinetic energy $c_K$ can be derived from the square dispersion of the kinetic energy per particle $\sigma_K^2 = \langle K^2\rangle - \langle K\rangle^2$ obtained after several realizations of the microscopic dynamics as follows:

$$c_K = \frac{1}{N-1}\left\{N\sigma_K^2 - \sigma_m^2\right\} \equiv \frac{1}{N-1}C_K. \tag{16}$$

where $\sigma_m^2 = \frac{1}{4}\left\{\langle p^4\rangle - \langle p^2\rangle^2\right\}$ is the square dispersion of the kinetic energy of an individual rotator, which in an analogous way as $\langle K\rangle$ only depends on the one-body distribution function. The short-time evolutions of the

[1] A regime with $\Delta_\theta(t) \propto t^\alpha$ exhibits a normal diffusion when $\alpha = 1$, ballistic diffusion $\alpha = 2$, and *anomalous diffusion* when $1 < \alpha < 2$ (superdiffusion) or $0 < \alpha < 1$ (sub-diffusion).

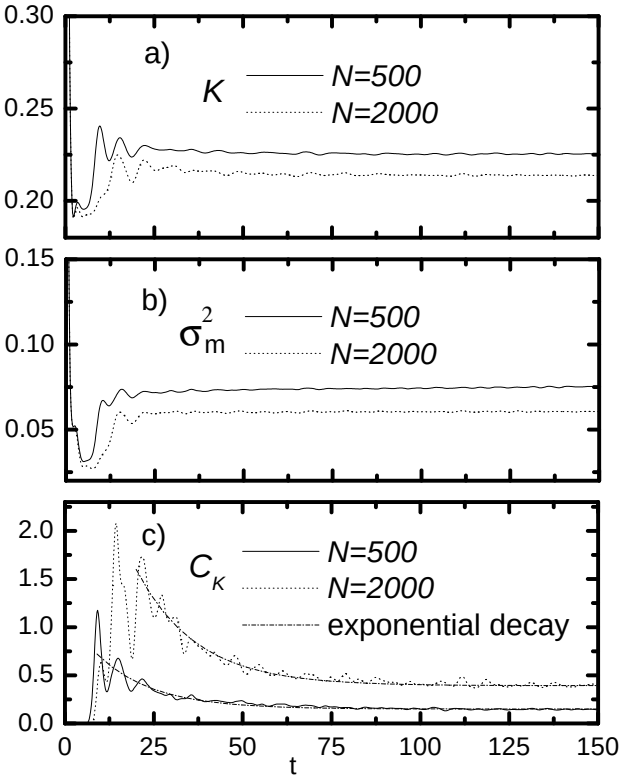


FIG. 5: Short-time evolutions of (a) the kinetic energy $\langle K\rangle$, (b) the square dispersion of individual rotators $\sigma_m^2$ and (c) the correlation function $C_K$ obtained from a statistical average over 1500 trajectories with $N = 500$ and 2000. The exponential decay of the correlation function $C_K$ evidences that the two-body correlation function $g(x_1, x_2; t)$ undergoes *a relaxation process in the same timescale of the violent relaxation* of the one-body distribution function $f(x; t)$.

functions $\langle K\rangle$, $\sigma_m^2$ and $C_K$ obtained from the average of microscopic dynamics with $N = 500$ and 2000 over 1500 trajectories are displayed in FIG.5.

While $\langle K\rangle$ and $\sigma_m^2$ account for the violent relaxation undergone by the one-body distribution function $f(\theta, p; t)$ during the first stages of the system evolution, the correlation function $C_K$ evidences also the *occurrence of a relaxation process for the two-body correlation function* $g(\theta_1, p_1, \theta_2, p_2; t)$ *in the same timescale*, where any two uncorrelated particles at the initial state become correlated after the incidence of such a relaxation process. Despite the existence of finite size effects, numerical simulations with different $N$ confirm the consistence of this observation. We shall show in a forthcoming paper that this result can be easily justified in terms of the well-known *BBGKY hierarchy*. Although simple, this latter behavior has a remarkable importance in understanding the collisional relaxation of this model system and the development of appropriate kinetic equations for the one-body distribution function $f(\theta, p; t)$.

### C. Dynamical phase coexistence

It is well-known that the existence of the long-range order leads to a very slow relaxation in the neighborhood of the critical point. However, most of works devoted to the numerical study of dynamical aspects of the HMF model also revealed the existence of very large relaxation times during the superdifussional regime within the energetic region $u_1 = 0.5 < u \leq u_c = 0.75$ [4, 5, 6, 7, 8, 9, 10, 12, 13]. It is remarkable that the above energetic region is precisely the same one where the microcanonical susceptibility $\chi_m(u; b = 0)$ experiences a large increasing below of the critical point $u_c$ (see in FIG.2 of the previous paper [19]). This observation suggests that the microcanonical results and these anomalous dynamical behaviors could be closely related.

While the thermodynamical characterization of this model shows the features of the typical second-order phase transitions, from the viewpoint of its out-of-equilibrium dynamics, the HMF model also shows certain analogy with the *phase coexistence phenomenon*. The present interpretation is naturally arisen from characterization of the rotator individual dynamics as a mathematical pendulum motion: when the pendulum energy $\varepsilon(\theta, p) = \frac{1}{2}p^2 - m\cos\theta < m$, the particle exhibits an oscillatory regime around the equilibrium configuration, while for $\varepsilon(\theta, p) > m$ the particle performs rotations in a given direction. The first kind of motion characterizes the particles trapped in the cluster existing when $u < u_c$, while the second one characterizes those particles that are not bonded to the cluster, which is the typical motion predominating in the homogeneous phase with $u > u_c$. We can rephrase the microscopic dynamics by considering that all those rotators satisfying the condition $\varepsilon(\theta, p) < m$ belong to a *dynamical clustered phase*, while the other with $\varepsilon(\theta, p) > m$ belong to a *dynamical gaseous phase*. Such a reinterpretation suggests the occurrence of a *dynamical phase coexistence* within the energetic region $u_1 < u < u_c$.

The relative population $p$ of the clustered phase can be obtained from the distribution function $f(\theta, p; t)$ as follows:

$$1 - p(t) = \int_m^\infty d\varepsilon F(\varepsilon; m, t), \tag{17}$$

where $F(\varepsilon; m, t)$ is the energy distribution function:

$$F(\varepsilon; m, t) = \int_0^{2\pi} d\theta \int_{-\infty}^{+\infty} dp\delta\{\varepsilon - \varepsilon(\theta, p)\} f(\theta, p; t). \tag{18}$$

Since during collisional regime the distribution function exhibits a quasi-stationary evolution throughout stable QSSs of Vlasov equation (7), the distribution function can be taken as $f_{QSS}(\theta, p; t) \simeq f[\varepsilon(\theta, p); m, t]$, which leads to the following form of the quasi-stationary energy distribution function:

$$F(\varepsilon; m, t) = T(\varepsilon, m) f(\varepsilon; m, t), \tag{19}$$

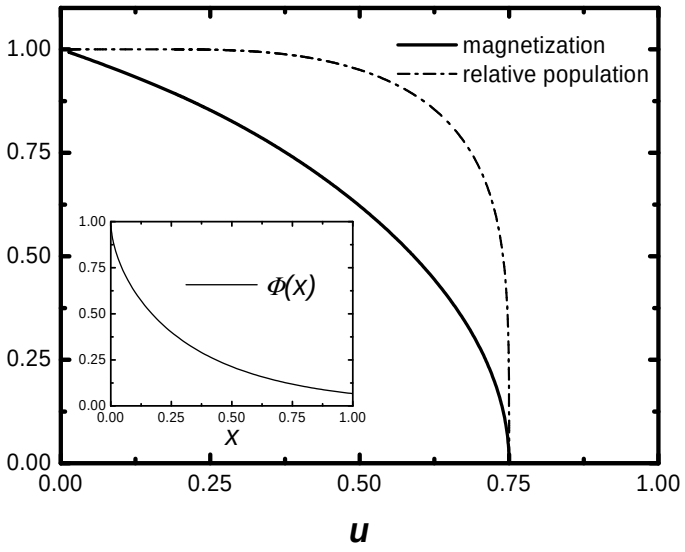


FIG. 6: Energy dependence of the relative population $p$ of the clustered phase and the magnetization $m$ (added for comparative purposes). Inserted graph: the $x$ dependence of the function $\Phi(x)$. This figure shows that the relative population of the clustered phase exhibits its most significant variation within the energy interval $u_1 < u \leq u_c$.

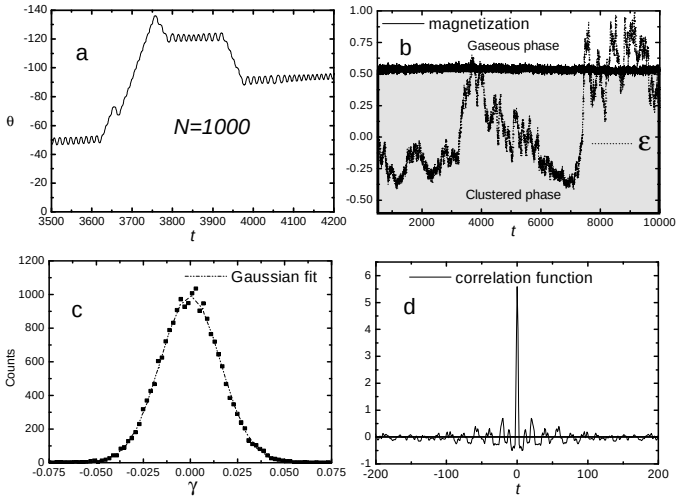


FIG. 7: "Particle interchange" between the clustered and the gaseous phases. Panel a: Transition between vibrational and rotational motions during the quasi-stationary regime; Panel b: Evolution of the energy $\varepsilon(t)$ of a given particle and the magnetization $m(t)$; Panels c and d: Frecuency counts and correlation function of the rate of change of the particle energy $\gamma(t) = d\varepsilon(t)/dt$.

being $T(\varepsilon, m)$ the time period of the mathematical pendulum with energy $\varepsilon$:

$$T(\varepsilon, m) = 4\int_0^{\theta_m} \frac{d\theta}{p(\theta;\varepsilon,m)} = 4\int_0^{\theta_m} \frac{d\theta}{\sqrt{2(\varepsilon + m\cos\theta)}}, \tag{20}$$

where $\theta_m$ is the positive turning point where $p(\theta_m;\varepsilon,m) = 0$ when $\varepsilon < m$, or $\theta_m \equiv \pi$ when $\varepsilon > m$.

The presence of the time period function $T(\varepsilon, m)$ in Eq.(19) demonstrates the existence of a *pole* in quasi-stationary energy distribution function at $\varepsilon = m$ ($\lim_{\varepsilon\to m} T(\varepsilon, m) = \infty$), which separates the energetic range of the system from the dynamical viewpoint in a clustered phase when $\varepsilon < m$ and a gaseous phase for $\varepsilon > m$. The incidence of such a lost of analyticity on the quasi-stationary regime is much significant for energies $u \leq u_c$ close to the critical point. The relatively large collisional relaxation times and the anomalous superdiffusional observed in the interval $u_1 < u < u_c$ should be related in some way to this dynamical phase coexistence. Unfortunately, a complete study of these question demands the knowledge of a kinetic equation describing the collisional evolution of the system.

Nevertheless, it could be useful to calculate the relative population $p$ of the clustered phase during the thermodynamic equilibrium of the HMF model. The integration of (17) by using the Boltzmann-Gibbs distribution function $f_{BG}(\varepsilon) \propto \exp(-\beta\varepsilon)$ yields:

$$p(x) = 1 - \frac{\exp(x)\,\Phi(x)}{I_0(x)}, \tag{21}$$

where $x = \beta m$, $I_0(x)$ the modified Bessel function of zero-order and $\Phi(x)$ is given by:

$$\Phi(x) = \frac{4}{\pi}\sqrt{\frac{2x}{\pi}}\int_0^1 \exp\left(-\frac{2x}{y^2}\right) K(y)\,\frac{dy}{y^2}, \tag{22}$$

being $K(x)$ the complete Legendre elliptic integral of the first kind:

$$K(x) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - x^2\sin^2 t}}. \tag{23}$$

The energetic dependence of the relative population $p$ of the clustered phase is illustrated in FIG.6. As expected, the system experiences an abrupt formation of the clustered phase below of the critical point $u_c$. Interestingly, the transformation of the clustered phase into the gaseous one mainly takes place on the anomalous energy interval $u_1 < u \leq u_c$, a result that reinforces the hypothesis about the origin of anomalous behavior in that region should be related to the occurrence of a dynamical phase coexistence during the out-of-equilibrium regime.

Panel a) in FIG.7 shows that a given rotator randomly changes the character of its motion during the quasi-stationary regime, a phenomenon which could be interpreted here as a "particle interchange" between the clustered and the gaseous phases. A better understanding of this process is achieved by analyzing the evolution of the particle energy $\varepsilon = \varepsilon(\theta, p) = \frac{1}{2}p^2 - \mathbf{m}\cdot\mathbf{m}(\theta)$ and the magnetization $m = |\mathbf{m}|$ illustrate in panel b) of this figure. Transitions towards the gaseous phase take place wherever the dependence $\varepsilon(t)$ overcome the magnetiza-

tion $m(t)$. Although the dynamical evolution of magnetization and the particle energy are actually smooth at the microscopic timescale $\tau_0$, these evolution look very rough at a larger timescale. It is very interesting to remark that the evolution of rate of change of the particle energy $\gamma(t) = d\varepsilon(t)/dt$ seems to be modeled as a *Gaussian noise* with a correlation time $\tau_c$ comparable to the microscopic timescale $\tau_0$, as shown in panels c) and d).

## III. PHENOMENOLOGICAL APPROACH OF COLLISIONAL RELAXATION

Recently, some authors are carrying out some pioneering studies in order to obtain some kinetic equations able to explain how this model system evolutes towards the final equilibrium configurations [14, 15, 16, 17, 18]. While the early relaxation dynamics is very-well described by the Vlasov dynamics (7), no one of such attempts satisfactorily explains the dynamical behavior of the HMF model in during the collisional regime.

This aim could be achieved by specifying the dynamical mechanism allowing the energetic interchange among the system particles during the collisional regime. As already commented in the previous section, the energetic interchange among the rotators during the collisional regime takes place as consequence of small oscillations or fluctuations of the magnetization vector $\mathbf{m}$ around the mean value $\langle \mathbf{m} \rangle$. The characteristic timescale of such fluctuations is comparable to the microscopic timescale $\tau_0$, which is the characteristic timescale of the rotator individual evolution. This latter observation and the form of Eq.(3) support the idea that the energy interchange in the HMF model takes place by means of the *parametric resonance* [21, 22, 23, 24, 25].

The mechanism of parametric resonance potentially appears whenever exist an oscillatory dependence of the microscopic parameters of an oscillatory system [21]. According with the common understanding, this phenomenon can manifest in the dynamics (3) when a Fourier mode $\omega$ of the magnetization $\mathbf{m}(t) = \langle \mathbf{m} \rangle + \sum_\omega \mathbf{a}_\omega(t) \cos(\omega t + \delta_\omega)$ belongs to a very small neighborhood of the frequency $\omega_i = 2\pi/T_i$ of a given rotator: $|\omega - \omega_i| \leq |\mathbf{a}_\omega| / \omega_0 \sim 1/\sqrt{N}$, being $\omega_0 \simeq \sqrt{\langle m \rangle}$.

At low energies, most of the system rotators exhibit a harmonic oscillatory motion whose frequencies are very close to $\omega_0$. Under these conditions the parametric resonance allows an affective energy interchange among the particles leading the system to a fast equilibration, which explains the concordance of thermodynamical predictions with the microcanonical numerical experiments at low energies [8]. The growing of the energy per particle leads to the increasing of the non harmonic character of the rotator motion, and consequently, the spreading of the Fourier modes of $\mathbf{m}(t)$. Thus, the necessary conditions for the parametric resonance are only eventually satisfied, allowing in this way to a given rotator to gain or lost a certain amount of energy, sometimes significant, in a brief time period, as already illustrated in FIG.7. Such large energetic transfers are rather analogous to a close encounter.

Recently studies about the origin of the Hamiltonian chaos by using the methods derived from the Riemannian interpretation of dynamics conclude that the phenomenon of parametric resonance is the fundamental mechanism to induce irregular motion in most of the Hamiltonian systems which are binding for most of regions of the configurational space [22, 23, 24, 25]. Apparently, the parametric resonance is the dynamics mechanics explaining the origin of dynamics instability and the collisional relaxation of the HMF model.

When the dynamical behavior of the system parameters is very complex, the treatment of such a microscopic dynamics could be performed by using the methods of theory of *random flights* (see in ref.[24] the application of such methods in the framework of the Hamiltonian chaos). This idea follows from the observation that the small oscillatory behavior of the magnetization vector in timescale longer than the microscopic time $\tau_0$ may be taken into account as a *Gaussian noise*. In fact, the dynamical evolution of the rotator energy $\varepsilon = \frac{1}{2}p^2 - \mathbf{m} \cdot \mathbf{m}(\theta)$ illustrated in FIG.7 suggests that the rotator dynamics is rather analogous to a *Brownian motion*, that is, a diffusion-like process.

Let us apply the present hypothesis in order to obtain a phenomenological kinetic equation for describing the collisional evolution of the HMF model. Hereafter, the Einstein summation convention is assumed. The magnetization vector $\mathbf{m}$ could be decomposed by considering the its mean value $\langle \mathbf{m} \rangle$ plus a *Gaussian noise* term as follows:

$$\mathbf{m} \simeq \langle \mathbf{m} \rangle + \mathbf{n}_a \eta_a, \tag{24}$$

where the unitary vectors $\mathbf{n}_a$ consider a different behavior of the noises $\eta_a$ in the parallel and transverse direction of the mean value $\langle \mathbf{m} \rangle$. Let us also assume that the noise only depends on the global system observables (system size $N$, energy $u$, average magnetization vector $\langle \mathbf{m} \rangle$, etc.). Since a purely diffusive influence does not lead the one-body distribution function towards the Gaussian profiles of velocities resulting at the final equilibrium, we shall take into consideration the inclusion of a *dynamical friction* in this context in order to accomplish this feature [26].

Thus, we conjecture that the microscopic dynamics of a given rotator could be conveniently approximated by the following Langevin equation with a *multiplicative noise*:

$$\dot{\theta} = p, \;\; \dot{p} = -\lambda(\theta)\, p + \mathbf{m}(\theta) \times \langle \mathbf{m} \rangle + e_a(\theta)\, \eta_a. \tag{25}$$

where the dynamical friction $\lambda(\theta)$ should necessarily be $\theta$-dependent (see in Eq.(30) below). The functions $e_a(\theta)$ are derived from the unitary vectors $\mathbf{n}_a$ as $e_a(\theta) = \mathbf{m}(\theta) \times \mathbf{n}_a$. The Gaussian noises $\eta_a$ satisfy the following relations:

$$\langle \eta_a(t) \rangle = 0, \;\; \langle \eta_a(t)\, \eta_b(t') \rangle = 2\Omega_{ab}\delta(t-t'). \qquad (26)$$

where $\Omega_{ab}$ is the correlation matrix. The Langevin equation (25) leads to the following Fokker-Planck equation:

$$\frac{\partial f}{\partial t} + p\frac{\partial f}{\partial \theta} + \mathbf{m}(\theta) \times \mathbf{m}[f] \frac{\partial f}{\partial p} = D[\theta] \frac{\partial}{\partial p}\left[\frac{\partial f}{\partial p} + \eta p f\right], \qquad (27)$$

where the mean value of the magnetization vector $\langle \mathbf{m} \rangle$ was taken as follows:

$$\langle \mathbf{m} \rangle = \mathbf{m}[f] = \int d\theta dp \; \mathbf{m}(\theta) f, \qquad (28)$$

and the diffusion coefficient $D[\theta]$ by:

$$D[\theta] = \Omega_{ab} e_a(\theta) e_b(\theta). \qquad (29)$$

We also assume that $D[\theta]$ is related to the dynamical friction by means of the Einstein relation:

$$\lambda(\theta) = \eta D(\theta), \qquad (30)$$

being $\eta$ the nonequilibrium analogue of the inverse temperature, $\eta^{-1} \equiv \int p^2 f(\theta, p; t)\, dpd\theta$. The consistence of Fokker-Planck equation (27) demands that the correlation matrix $\Omega_{ab}$ should be considered as certain functionals of the distribution function $f$. Since $\Omega_{ab}$ decreases with the increasing of the system size, the diffusive term of Fokker-Planck equation is only effective at large temporal scales. Thus, the assumption (28) takes into account that the dynamical evolution of the distribution function at the timescale $\tau_0$ is described by the collisionless dynamics (7), while diffusive term considers the relaxation towards the Boltzmann-Gibbs distribution function at the collisional timescale $\tau_{cr} \propto D^{-1}$.

The definition of the diffusion coefficient $D(\theta)$ can be conveniently rewritten by considering the *tridimensional* vector $\mathbf{m} = (\cos\theta, \sin\theta, 0)$. Thus, the scalar term $e_a e_b$ could be rephrased as:

$$e_a e_b \equiv (\mathbf{m} \times \mathbf{n}_a) \cdot (\mathbf{m} \times \mathbf{n}_b) = \delta_{ab} - (\mathbf{m} \cdot \mathbf{n}_a)(\mathbf{m} \cdot \mathbf{n}_a). \qquad (31)$$

Let us now reconsider again the bidimensional form of the vector $\mathbf{m}(\theta) = (\cos\theta, \sin\theta)$. By denoting the $i$-th component of $\mathbf{m}(\theta)$ by $m_i(\theta)$, the diffusion coefficient can be expressed as:

$$D = m_i(\theta) \mathcal{K}_{ij} m_j(\theta), \qquad (32)$$

where $\mathcal{K}_{ij} = \Omega\delta_{ij} - \Omega_{ab}(n_a)_i (n_b)_j$, and $\Omega = Sp[\Omega_{ab}]$. Since the only preferential direction of this system is the mean value of the magnetization $\langle \mathbf{m} \rangle$, the symmetric matrix $\mathcal{K}_{ij}$ should take the form:

$$\mathcal{K}_{ij} = a[f]\delta_{ij} + b[f] \langle m_i \rangle \langle m_j \rangle, \qquad (33)$$

allowing to rewrite the diffusion factor as follows:

$$D(\theta) = a[f] + b[f] \{\mathbf{m}[f] \cdot \mathbf{m}(\theta)\}^2. \qquad (34)$$

The closure of the Fokker-Planck approximation relies on the determination of the functional $a[f]$ and $b[f]$, which is a task beyond of our phenomenological reasonings. It is very easy to verify that Eq.(27) ensures the particles, momentum and the energy conservation.

Chavanis *et al* introduced in refs.[14, 15, 16] a similar treatment by modifying of the HMF model microscopic dynamics (2) with the inclusion of frictional and stochastic terms, whose resulting Fokker-Planck equation basically differs from the present proposal by the $\theta$-dependence of the diffusion coefficient $D(\theta)$. The origin of this difference is found in the fact that these authors actually introduce a variant of HMF model (Brownian Mean Field model) which accounts for the external influence of a thermostat, while the frictional and stochastic terms considered in the present work try to capture the effective influence of the magnetization fluctuations on the exact individual dynamics of a given rotator in a timescale larger than the microscopic time $\tau_0$. The crucial question now is how to justify and precise within the framework of the parametric resonance the *ad hoc* approximation (25), mainly, the inclusion of the effective dynamical friction.

## IV. FINAL REMARKS

As already appreciated in many studies, the HMF model dynamics undergoes the incidence of two relaxation process with different characteristic timescales: (1) the violent relaxation timescale $\tau_{vr}$ leading to the quasi-stationary evolution with a superdiffusion regime, and (2) the collisional relaxation timescale $\tau_{cr}$ where the Boltzmann-Gibbs equilibrium is achieved as final state. It is well-established theoretically and numerically that the collisionless relaxation timescale is the same order of the microscopic timescale, $\tau_{vr} \sim \tau_0$, while the collisional relaxation timescale seems to grow with the system size in some power of $N$, $\tau_{cr} \sim \tau_0 N^{\alpha}$. Early estimations predict a simple linear dependence [9], although there are also several numerical experiments evidencing a nontrivial exponent $\alpha \simeq 1.7$ [10, 12, 13].

Generally speaking, the determination of the correct $N$-dependence of the collisional relaxation time is rather difficult to carry out by means of numerical computations of the microscopic dynamics since such experiments are plagued of prejudicial size effects and poor equilibration of trajectory averages whose elimination

seems to demand the consideration of more larger number of particles than the actual computational possibilities $N \sim 10^4 - 10^5$ [10]. For example, besides of the nontrivial exponent $\alpha = 1.7$ of the power-law growing of $\tau_{cr}$ reported by Yamaguchi *et al* in refs.[12, 13], they also indicated the following $N$-dependence of the quantity $q$:

$$q = \left. \frac{d \langle m(t) \rangle}{d(\ln t)} \right|_{t=\tau_{cr}} \propto \sqrt{N}. \qquad (35)$$

If the above estimation of $\tau_{cr}$ is the correct relaxation timescale associated to the collisional quasi-stationary evolution (qse) of the one-body distribution function $f(\theta, p; t)$, the temporal dependence for large $N$ should be given by:

$$f(\theta, p; t) \simeq f_{qse}(\theta, p; t/\tau_{cr}) + O(1/N). \qquad (36)$$

The above assumption implies that the leading behavior the temporal dependence of the average magnetization should be given by $\langle m(t) \rangle \simeq A(t/\tau_{cr})$, where $A(s) = \left| \int \mathbf{m}(\theta) f_0(\theta, p; s) \, d\theta dp \right|$ is a size-independent function. Thus, the leading behavior of the quantity:

$$q = \left. \frac{d \langle m(t) \rangle}{d \ln t} \right|_{t=\tau_{cr}} \equiv \frac{dA(s=1)}{ds}, \qquad (37)$$

*should be N-independ* when $N$ is large enough. Since the numerical result (35) disagrees with this reasoning, the nontrivial exponent $\alpha \simeq 1.7$ reported in the Yamaguchi *et al* study might not correspond to the true scaling behavior for $N$ very large, but only a finite size behavior which should disappear for $N$ large enough.

The size dependence of the collisional relaxation timescale is a question with a primordial importance in order to understand the dynamical features of a long-range interacting system as the HMF model when thermodynamic limit is invoked. As already pointed out by some authors, the existence of relaxation timescale $\tau_{cr}$ diverging with the imposition of the thermodynamic limit, $\lim_{N\to\infty} \tau_{cr} = \infty$, leads automatically to the non commutative character of the the thermodynamic limit $\lim_{N\to\infty}$ with the infinite time limit $\lim_{t\to\infty}$ necessary to the equilibration of temporal averages $\langle A_N \rangle_t = \int_0^t A_N(\tau) \, d\tau / t$:

$$\lim_{N\to\infty} \lim_{t\to\infty} \frac{\langle A_N \rangle_t}{N} \neq \lim_{t\to\infty} \lim_{N\to\infty} \frac{\langle A_N \rangle_t}{N}, \qquad (38)$$

Thus, the imposition of the thermodynamic limit before the infinite time limit makes endless the duration of collisionless dynamics, and therefore, the Boltzmann-Gibbs equilibrium arises as an admissible but particular quasi-stationary state of the violent relaxation. Such a dynamical picture is not only a feature of the HMF model dynamics. On the contrary, the large the collisional relaxation times estimates of many astrophysical objects like the elliptical galaxies ($\tau_{cr} \sim 0.1 \tau_0 N / \ln N$ where microscopic timescale $\tau_0 \sim 1/\sqrt{G\rho}$) induces also to suppose the collisionless character of the dynamical evolution of such real long-range interacting systems, which explains the rich variety of structures observed in this context [20].

Let us use the phenomenological approach of collisional regime in terms of the Fokker-Planck equation (27) in order to analyze the $N$-dependence of the collisional relaxation timescale $\tau_{cr}$. Since the amplitude of the Gaussian noises $\eta_a$ describing the magnetization fluctuations during the quasi-stationary evolution decreases as $1/\sqrt{N}$ as well as the underlying correlation times are of order of microscopic timescale $\tau_0$ (see in panel c) of FIG.7), the correlation matrix $\Omega_{ab}$ (26) and the diffusion coefficient $D(\theta)$ (29) decrease as $1/N$. Thus, the phenomenological picture described in this work suggests that the relevant timescale of the collisional evolution of the HMF model should obey a linear growing $\tau_{cr} = \tau_0 N$.

As already discussed in our previous work [19], such a linear $N$-dependence of $\tau_{cr}$ seems to be intimately related to the $N$-dependence of the additive constant of entropy per particle $s_0 = \frac{1}{2} \ln \left( 2\pi e^2 I g / N \right)$ , since the imposition of a scaling dependence of the coupling constant $g$ as $g(N) = \gamma N$ instead of the usual Kac prescription [27] $g_{kac}(N) = \gamma / N$ leads to a *simultaneous regularization* of the divergence of the collisional relaxation timescale $\tau_{cr} = \tau_0 N = \sqrt{IN/g} \equiv \sqrt{I/\gamma}$ and the additive constant $s_0 = \frac{1}{2} \ln \left( 2\pi e^2 I g / N \right) \equiv \frac{1}{2} \ln \left( 2\pi e^2 I \gamma \right)$ in the thermodynamic limit [19], avoiding in this way the incidence of an undesirable dynamical anomalies like the one described in Eq.(38) and the divergence of the thermodynamic potentials.

Although the phenomenological approach of collisional relaxation described in this work supports the validity of the linear growing $\tau_{cr} = \tau_0 N$, the problem about the exact $N$-dependence of collisional relaxation timescale could be definitively solved by means of the development of appropriate kinetic equations starting from the consideration of first principles [16]. We shall address in our forthcoming paper an intense investigation of the HMF model dynamics in terms of kinetic equations, where the implementation of suitable kinetic equations starting from the well-known *BBGKY hierarchy* will receive a primordial attention.


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